

EXACT PROPERTIES OF FROBENIUS NUMBERS AND FRACTION OF THE SYMMETRIC SEMIGROUPS IN THE WEAK LIMIT FOR $N=3$

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ABSTRACT. We generalize and prove a hypothesis by V. Arnold on the parity of Frobenius number. For the case of symmetric semigroups with three generators of Frobenius numbers we found an exact formula, which in a sense is the sum of two Sylvester's formulae. We prove that the fraction of the symmetric semigroups is vanishing in the weak limit.

1. DEFINITIONS

Take n mutually prime numbers $a_1, a_2, \dots, a_n : (a_1, a_2, \dots, a_n) = 1$. Consider semigroup $S = \{s = x_1 a_1 + \dots + x_n a_n \mid x_i \in \mathbb{Z}_+\}$.¹ It follows that starting from some number $F(a_1, a_2, \dots, a_n) \in S$ all integers are in the set S :

Definition 1. $F(a_1, a_2, \dots, a_n) = \min(s \in S \mid \forall k \in \mathbb{Z}_+, k \geq s : k \in S)$.

Number $F(a_1, a_2, \dots, a_n)$ is called a Frobenius number² and its properties are the subject of the paper. For $n = 2$ the exact formula is known, the Sylvester formula [1], for F : $F(a_1, a_2) = (a_1 - 1)(a_2 - 1)$. It is known [2] that for an arbitrary relatively prime set (a_1, a_2, \dots, a_n) , $n \geq 3$ the Frobenius number cannot be expressed in terms of a finite set of polynomials.

There are two useful functions. Function $C(a_1, a_2, \dots, a_n) = F(a_1, a_2, \dots, a_n) - 1$ is the maximal integer which is not in the set S . Function $G(a_1, a_2, \dots, a_n) = F(a_1, a_2, \dots, a_n) - 1 + a_1 + a_2 + \dots + a_n$ is an analog of the function C for the semigroup $S_1 = \{s = x_1 a_1 + \dots + x_n a_n \mid x_i \in \mathbb{N}\}$.

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¹In the paper, \mathbb{N} is the set of all positive integers, and \mathbb{Z}_+ is the set of all non-negative integers.

²We have to note that often the Frobenius number defined as a number less by unity (compare with the function C defined below). In the paper we will keep given notation which is also used in papers by V. Arnold.

Note, that condition $s < F$ and $s \in S$ leads to the result that $C - s \notin S$ (otherwise, $C \in S$). The opposite statement is not true in the general case.

Definition 2. Semigroup S is called symmetric if $\forall s < F, s \notin S : C - s \in S$.

Each semigroup generated by two elements is the symmetric one.

2. EXACT RESULTS

Analyzing properties of Frobenius numbers and associated semigroups, V. Arnold noticed that if two of three generating elements are even then F is even as well [3]. The following Theorem is the generalization of the experimental observation and gives the proof. Note, that renumbering of the generating elements does not change the structure of the set S and, in particular, does not change value of F . For the sake of simplicity we shall use this property in following without explicitly mentioning it.

Theorem 1. Let us $(a_1, a_2, \dots, a_{n-1}, a_n - 1) = d$. Then F is divisible by d .

Proof. By the definition of F : $F - 1 + a_n \in S(a_n \geq 1)$. Hence, there exist nonnegative integers x_1, \dots, x_n such that $F - 1 + a_n = x_1 a_1 + \dots + x_n a_n$. Therefore, $F - 1 = x_1 a_1 + \dots + x_{n-1} a_{n-1} + (x_n - 1) a_n$. If $x_n \geq 1$, then $F - 1 \in S$, what contradicts to the definition of F . Hence, $x_n = 0$. $F = x_1 a_1 + \dots + x_{n-1} a_{n-1} - (a_n - 1)$. Thus, F is divisible by d . ■

Let $d_{ij} = (a_i, a_j), i \neq j$.

Lemma 1. Johnson's lemma [4]. $G(a_1, a_2, a_3) = d_{12} G\left(\frac{a_1}{d_{12}}, \frac{a_2}{d_{12}}, a_3\right)$.

Lemma 2. Brauer-Shockley lemma [5]. Let $d = (a_1, a_2, \dots, a_{n-1})$. Then, $C(a_1, a_2, \dots, a_n) = dC(a_1/d, \dots, a_{n-1}/d, a_n) + (d - 1)a_n$.

Divisibility of F by d follows also ³ from the Lemma 2. Indeed, $F(a_1, \dots, a_n) = C(a_1, \dots, a_n) + 1 = dC(a_1/d, \dots, a_{n-1}/d, a_n) + (d - 1)a_n + 1 = d(C(a_1/d, \dots, a_{n-1}/d, a_n) + a_n) - (a_n - 1)$.

Lemma 3. Semigroup $S(a_1, a_2, a_3)$ is symmetric if and only if for some i : $a_i \in S\left(\frac{a_j}{d_{jk}}, \frac{a_k}{d_{jk}}\right)$, where $i \neq j \neq k$.

In the case of symmetric semigroups we can prove the following exact formula for the Frobenius numbers with $n = 3$. Denote $b_l = \frac{a_l}{d_{12}}, l = 1, 2$. It is always possible to renumerate generating elements in such a way that under the conditions of Lemma 2 it takes place that $i = 3$.

Theorem 2. Let $S(a_1, a_2, a_3)$ is a symmetric semigroup, $a_3 \in S(b_1, b_2)$. Then $F(a_1, a_2, a_3) = d_{12} F\left(\frac{a_1}{d_{12}}, \frac{a_2}{d_{12}}\right) + F(d_{12}, a_3)$.

³We acknowledge to referee for this remark.

Proof. $F(a_1, a_2, a_3) = G(a_1, a_2, a_3) - a_1 - a_2 - a_3 + 1 =$ /by Lemma 1/ $= d_{12}G\left(\frac{a_1}{d_{12}}, \frac{a_2}{d_{12}}, a_3\right) - a_1 - a_2 - a_3 + 1 = d_{12}(F(b_1, b_2, a_3) + b_1 + b_2 + a_3 - 1) - d_{12}b_1 - d_{12}b_2 - a_3 + 1 = d_{12}F(b_1, b_2) + d_{12}a_3 - a_3 - d_{12} + 1 =$ /by Sylvester's formula/ $= d_{12}F(b_1, b_2) + F(d_{12}, a_3)$. Note, that $(d_{12}, a_3) = 1$ due to the $(a_1, a_2, a_3) = 1$. ■

Notice. If semigroup $S(a_1, a_2, a_3)$ is not symmetric then $F(a_1, a_2, a_3) < d_{12}F\left(\frac{a_1}{d_{12}}, \frac{a_2}{d_{12}}\right) + F(d_{12}, a_3)$, because if $a_3 \notin S(b_1, b_2)$ then $F(b_1, b_2, a_3) < F(b_1, b_2)$.

At the time of the paper revision the author found in the literature theorem in Chinese language [6] similar to our Theorem 2.

3. ASYMPTOTIC PROPERTIES

Theorem 2 shows that, informally speaking, if a_3 is sufficiently large ($a_3 \geq F(b_1, b_2)$), then $F(a_1, a_2, a_3)$ behaves as a Frobenius number with two generating elements (compare with the Sylvester formula!). Therefore, it is interesting to consider the case under which all numbers a_1, a_2, a_3 are of the same order in some sense. V. Arnold [7] proposed to investigate an asymptotic behavior in the following way. Let us fix vector $\mathbf{a} = (a_1, \dots, a_n)$ and for some function $f(\mathbf{a})$ consider the following average:

$$V(f, \mathbf{a}, N) = \sum_{-r \leq r_i \leq r, i=1, \dots, n} f(N\mathbf{a} + \mathbf{r}) / (2r)^n,$$

where $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$ and $r, N \in \mathbb{N}$. Let us choose set $r(N)$ such that

$$(1) \quad r(N) \rightarrow \infty, \frac{r(N)}{N} \rightarrow 0 \text{ if } N \rightarrow \infty.$$

Let us consider the limit:

$$L(f, \mathbf{a}) = \lim_{r \rightarrow \infty, N \rightarrow \infty, r/N \rightarrow 0} (V).$$

Let $Sym(\mathbf{a}) = \begin{cases} 1 & \text{if } S(a_1, a_2, a_3) \text{ - symmetric} \\ 0 & \text{- otherwise} \end{cases}$. Fel [8] gives

theorem which in our terminology sounds as follows.

Theorem 3. $L(Sym, \mathbf{a}) = 0$.

Unfortunately, proof given in [8] contains some mistakes. For example, author says that set of triples $(a_1N + r_1, a_2N + r_2, a_3N + r_3)$, $|a_i| \leq r(N)$, $i = 1, 2, 3$, which constitute symmetric semigroups and such that $D = (a_1N + r_1, a_2N + r_2) \sim N$, is empty while $N \rightarrow \infty$. Let us give counter example. Let $a_1 = 4, r_1 = 2, a_2 = 6, r_2 = 3$. Then $D = 2N + 1 \sim N$ and $F(\frac{a_1N+r_1}{D}, \frac{a_2N+r_2}{D}) = 2$, i.e. for any

set $r(N)$, fulfilling (1), and any a_3 for large enough N semigroup $S(a_1N + r_1, a_2N + r_2, a_3N + r_3)$ (if exists) is symmetric. Thus, considered set are not always empty. We present here some other proof of the Theorem 3. The main idea is not to prove an emptiness but rather prove the fact that the fraction of symmetric semigroups is small in the considered limit.

We say that the fraction of the symmetric semigroups is vanishing in the weak asymptotics. Fel showed [8] that in the limit of interest the fraction of the mutually simple numbers (those sets on which Frobenius numbers defined) is equal $\frac{1}{\zeta(n)}$. In other words for

$$I(\mathbf{a}) = \begin{cases} 1 & \text{if } (a_1, \dots, a_n) = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{we have } L(I, \mathbf{a}, 1) = \frac{1}{\zeta(n)}.$$

Proof. Let $A_i = Na_i + r_i$. Let us figure out the number of semigroups in the cubic r -vicinity of the point $N\mathbf{a}$ those for which $A_3 \in S_0 = S\left(\frac{A_1}{D}, \frac{A_2}{D}\right)$, where $D = (A_1, A_2)$. Note, that for sufficiently large D Frobenius number $F\left(\frac{A_1}{D}, \frac{A_2}{D}\right)$ is not larger then A_3 , therefore for sufficiently large D any A_3 lies in semigroup S_0 . The number of semigroups such that $D \geq D_0$ (value of D_0 will be chosen later on) may be estimated as:

$$U_1 = \sum_{D=D_0}^{cN} \left(\frac{2r}{D}\right)^2 \leq 4r^2 \int_{D_0-1}^{cN-1} \frac{1}{x^2} dx = 4r^2 \left(\frac{1}{D_0-1} - \frac{1}{cN} \right),$$

where constant c may be chosen as $c = \min(a_1, a_2) + 1$ because if $N \rightarrow \infty, r \rightarrow \infty, r/N \rightarrow 0$ then $D \leq \min(A_1, A_2) \leq cN$.

Now consider the case $D < D_0$. Let us fix some $\varepsilon > 0$. Then $\exists N_0$ such that $\forall N > N_0$ we have $r(N) < \varepsilon N$, i.e. $\forall N > N_0 \forall r_3: |r_3| < r(N)$ we have that $a_3N + r_3 < (a_3 + \varepsilon)N$. Denote $c_1 = a_3 + \varepsilon$. Let us estimate the number of elements of the semigroup $S\left(\frac{A_1}{D}, \frac{A_2}{D}\right)$ in the interval $[0; c_1N]$: $s = x_1a_1 + x_2 + a_2$. (See Fig. 1) Without lack of generality let that $a_1 < a_2$. Then for sufficiently large N $\frac{A_1}{D} < \frac{A_2}{D}$. Denote $T = \left\lfloor \frac{c_1ND}{A_1} \right\rfloor$. Notice that $x_1 + x_2 \leq T$. With fixed number x_2 number x_1 may take not more than $T - x_2$ values. Therefore,

$$\# \left(S\left(\frac{A_1}{D}, \frac{A_2}{D}\right) \cap [0; c_1N] \right) \leq \sum_{x_2=0}^T (T - x_2) = \sum_{k=0}^T k = \frac{T(T+1)}{2}.$$

Then with fixed A_1 and A_2 and any $A_3 = a_3N + r_3$, $|r_3| \leq r(N)$ the number of symmetric semigroups are not larger than $\left(S\left(\frac{A_1}{D}, \frac{A_2}{D}\right) \cap [0; c_1N] \right)$. So the number of symmetric semigroups such that $D < D_0$, may be

estimated as

$$U_2 \leq \sum_{D=1}^{D_0} \left(\frac{2r}{D} \right)^2 \left(C_1^2 \left(\frac{N}{A_1} \right)^2 D^2 + O(D) \right) \approx 4r^2 C_1^2 \left(\frac{N}{a_1 N + r_1} \right)^2 D_0.$$

Thus, the fraction of symmetric semigroups is estimated as

$$(2) \quad W = 3 \frac{U_1 + U_2}{8r^3} \leq 3 \left(\frac{1}{D_0 - 1} - \frac{1}{cN} + \frac{D_0}{2r} C_1^2 \left(\frac{N}{a_1 N + r_1} \right)^2 \right).$$

Let $D_0 = \ln r$. Then, with $N \rightarrow \infty, r \rightarrow \infty, r/N \rightarrow 0$ the right hand side of inequality (2) vanishes, i.e. $W \rightarrow 0$. ■

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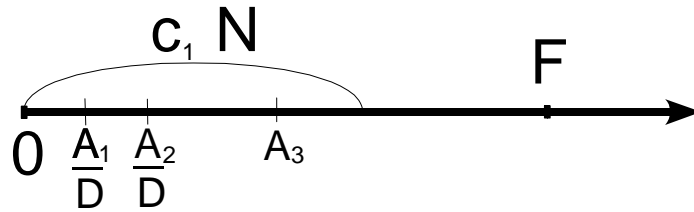


FIGURE 1.

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